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ITERATED COMMUTATORS UNDER A JOINT CONDITION ON THE TUPLE OF MULTIPLYING FUNCTIONS

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ABSTRACT. We present a pair of joint conditions on the two functions b_1, b_2 strictly weaker than $b_1, b_2 \in \mathrm{BMO}$ that almost characterize the L^2 boundedness of the iterated commutator $[b_2, [b_1, T]]$ of these functions and a Calderón-Zygmund operator T. Namely, we sandwich this boundedness between two bisublinear mean oscillation conditions of which one is a slightly bumped up version of the other.

1. INTRODUCTION

The study of commutators of Calderón-Zygmund operators with pointwise multiplication has been a long standing interest in the field of harmonic analysis; for example, in the fundamental paper of Coifman, Rochberg, Weiss [2] a characterization of the space $\mathrm{BMO}(\mathbb{R}^d)$ is given with respect to the commutators taken with the Riesz transforms:

$$[b, R_j] : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \quad \text{boundedly for all} \quad j = 1, \ldots, d$$

if and only if $b \in \mathrm{BMO}(\mathbb{R}^d)$. Here $[b, R_j] = bR_j - R_j(b\,\cdot)$. Already in [2] it was shown that $b \in \mathrm{BMO}$ is a sufficient condition for the boundedness of the iterated commutator $[b, [b, \ldots, [b, T]]]$ of pointwise multiplications and a Calderón-Zygmund operator and the same argument extends to the case of commutators $[b_k, [b_{k-1}, \ldots, [b_1, T]]]$ with different functions, all in BMO separately.

Our object is to make the first systematic study of the iterated commutator $[b_2, [b_1, R_j]]$ in the case of two different functions b_1, b_2. In particular, we want to identify a joint condition on the pair (b_1, b_2) that is weaker than the individual conditions $b_1, b_2 \in \mathrm{BMO}$, that is as close to optimal as possible, and which still guarantees the boundedness of the commutator. This is, in some sense, similar in spirit to the case of bilinear weighted theory, where $w_1, w_2 \in A_4$ is not the optimal condition for the boundedness of bilinear singular integrals from $L^4(w_1) \times L^4(w_2)$ to $L^2(w_1^{1/2} w_2^{1/2})$, but rather there is a genuinely bilinear joint condition $(w_1, w_2) \in A_{(4,4)}$ introduced by Lerner, Ombrosi, Pérez, Torres and Trujillo-González [11]. In the weighted case the identification of this genuinely bilinear condition has been highly impactful.

We study two-sided estimates for the $L^2 \to L^2$ norm of the commutator $[b_2, [b_1, T]]$. While the upper bounds will be valid for all bounded singular integrals, the lower bounds require some suitable non-degeneracy, and here we work with the Riesz transforms

$$R_j f(x) = \lim_{\varepsilon \to 0} \int_{|x-y| > \varepsilon} \frac{x_j - y_j}{|x-y|^{d+1}} f(y)\, \mathrm{d}y, \quad f \in L^2(\mathbb{R}^d), \quad j = 1, \cdots, d.$$

We show that

$$C_d(S_2(b_1, b_2) + T_2(b_1, b_2)) \leq \big\| [b_2, [b_1, T]] \big\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq C_{T,\varepsilon}(S_{2+\varepsilon}(b_1, b_2) + T_{2+\varepsilon}(b_1, b_2)), \quad (1.1)$$

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where the constant $C_{T,\varepsilon}$ tends to infinity as ε tends to zero and the joint conditions S_p, T_p, with $0 < p < \infty$, imposed on the complex valued functions b_1, b_2 are defined by

$$S_p(b_1, b_2) = \sup_Q \left(\frac{1}{|Q|} \int_Q |b_1 - \langle b_1 \rangle_Q|^p \right)^{1/p} \left(\frac{1}{|Q|} \int_Q |b_2 - \langle b_2 \rangle_Q|^p \right)^{1/p}$$

and

$$T_p(b_1, b_2) = \sup_Q \left(\frac{1}{|Q|} \int_Q |b_1 - \langle b_1 \rangle_Q|^p |b_2 - \langle b_2 \rangle_Q|^p \right)^{1/p}, \qquad b_1, b_2, b_1 b_2 \in L^p_{loc}(\mathbb{R}^d).$$

Here the supremums are taken over all cubes. Whenever it is well understood which functions b_1, b_2 are in question, we refer to these conditions shortly as T_p and S_p.

We show by example that the lower bound in (1.1) does not improve to $S_{2+\varepsilon}(b_1, b_2) + T_{2+\varepsilon}(b_1, b_2)$ for any $\varepsilon > 0$ – that is, the obtained upper bound is not necessary. This leads us to consider joint conditions involving Young functions that can be made strictly weaker than $S_{2+\varepsilon} + T_{2+\varepsilon}$ for all $\varepsilon > 0$. Hence, we prove the commutator upper bound with these updated conditions with a version of the sparse domination principle introduced in Lerner [9].

1.1. **Basic notation.** We denote $A \lesssim B$, if $A \leq CB$ for some constant $C > 0$ depending only on the dimension of the underlying space, on integration exponents and on other concurrently unimportant absolute constants appearing in the assumptions. Then naturally $A \sim B$, if $A \lesssim B$ and $B \lesssim A$. Subscripts on constants ($C_{a,b,c,\ldots}$) signify their dependence on those subscripts.

We also denote the space $L^p(\mathbb{R}^d)$ with L^p.

Integral average is by dash or brackets: $\frac{1}{|Q|} \int_Q f = \fint_Q f = \langle f \rangle_Q$.

When we say that an operator A is bounded on L^p we mean that $A : L^p \to L^p$ boundedly.

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2. Necessary conditions

We move on to derive the lower bound $S_2 + T_2$ for the iterated commutator taken with the Riesz transforms. Later we see that the condition $S_2 + T_2$ is not strong enough to imply the L^2 boundedness of the commutator, however.

Before proceeding any further, let us precisely define the commutator $[b_2, [b_1, T]]$.

Definition 2.1. Let b_i, $i = 1, 2$, be such that $b_1, b_2, b_1 b_2 \in L^2_{loc}(\mathbb{R}^d)$ and denote $b = (b_1, b_2)$. With T being an operator on $L^2(\mathbb{R}^d)$, the commutator $C_b T$ on $L^\infty_c(\mathbb{R}^d)$ is defined as

$$C_b T = [b_2, [b_1, T]],$$

where $[A, B] = AB - BA$ for any two operations A, B, and $b_i f(\cdot) = b_i(\cdot) f(\cdot)$.

We deal with the second order commutator $[b_2, [b_1, T]]$ but our results concerning sufficient conditions could just as well be formulated in the higher order cases.

Lemma 2.2. *Let R_j be the jth Riesz transform on $\mathbb{R}^d$, $j = 1, \ldots, d$, $f_1, f_2 \in L^\infty_c$ and $b_1, b_2, b_1 b_2 \in L^1_{loc}$. Under these assumptions, for all cubes Q, we have that*

$$\left| \fint_Q \fint_Q \prod_{i=1}^2 (b_i(x) - b_i(y)) f_2(y) f_1(x) \, \mathrm{d}y \, \mathrm{d}x \right| \leq C_d \sum_{i=1}^k \left\| C_b R_i \right\|_{L^p \to L^p} \left(\fint_Q |f_1|^p \right)^{1/p} \left(\fint_Q |f_2|^{p'} \right)^{1/p'}.$$

Proof. Our proof separates into two cases, to odd and even dimensions.

Case 1, d is odd: Let $d = 2k+1$ for some $k \in \mathbb{N}$. By composing back and forth with the translation $x \mapsto x - c_Q$, we may assume that the cube Q is centred at the origin. We begin with introducing 1 as

$$1 = \sum_{i=1}^d \frac{(x_i - y_i)^2}{|x - y|^2} = \sum_{i=1}^d \frac{(x_i - y_i)}{|x - y|^{d+1}} (x_i - y_i) |x - y|^{d-1}, \tag{2.1}$$

denote $b(x,y) = \prod_{i=1}^2 (b_i(x) - b_i(y))$, and then proceed with:

$$\begin{aligned}
&\Big| \int_Q \int_Q b(x,y) f_2(y) f_1(x) \,\mathrm{d}y\,\mathrm{d}x \Big| = \Big| \int_Q \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} b(x,y) f_2(y) 1_Q(y) f_1(x) \,\mathrm{d}y\,\mathrm{d}x \Big| \\
&= \Big| \int_Q \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} b(x,y) \sum_{i=1}^d \frac{(x_i-y_i)^2}{|x-y|^2} f_2(y) 1_Q(y) f_1(x) \,\mathrm{d}y\,\mathrm{d}x \Big| \\
&= \Big| \int_Q \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} b(x,y) \sum_{i=1}^d \frac{Y_i(x-y)^2}{|x-y|^{2d}} f_2(y) 1_Q(y) f_1(x) \,\mathrm{d}y\,\mathrm{d}x \Big|,
\end{aligned}$$

where $Y_i(x) = x_i|x|^{d-1}$. We momentarily force the expression into this form in order to contrast it with the similar argument emplying spherical harmonics given in [2].

For a given $\alpha = (\alpha_1,\ldots,\alpha_d) \in \mathbb{N}^d$ and $x \in \mathbb{R}^d$, let $x^\alpha = \prod_{i=1}^d x_i^{\alpha_i}$. We continue with

$$\begin{aligned}
&\Big| \int_Q \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} b(x,y) \sum_{i=1}^d \frac{Y_i(x-y)^2}{|x-y|^{2d}} f_2(y) 1_Q(y) f_1(x) \,\mathrm{d}y\,\mathrm{d}x \Big| \\
&= \Big| \int_Q \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} b(x,y) \sum_{i=1}^d \frac{x_i-y_i}{|x-y|^{d+1}} (x_i-y_i) \Big(\sum_{j=1}^d (x_j-y_j)^2 \Big)^k f_2(y) 1_Q(y) f_1(x) \,\mathrm{d}y\,\mathrm{d}x \Big| \\
&= \Big| \int_Q \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} \sum_{i=1}^d \sum_{\alpha+\beta=d} b(x,y) \frac{x_i-y_i}{|x-y|^{d+1}} a_{\alpha,\beta} f_1(x) x^\alpha y^\beta f_2(y) 1_Q(y) \,\mathrm{d}y\,\mathrm{d}x \Big| \\
&\overset{*}{=} \Big| \sum_{i=1}^d \sum_{\alpha+\beta=d} a_{\alpha,\beta} \int_Q f_1(x) x^\alpha \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} b(x,y) \frac{x_i-y_i}{|x-y|^{d+1}} y^\beta f_2(y) 1_Q(y) \,\mathrm{d}y\,\mathrm{d}x \Big| \\
&\le \sum_{i=1}^d \sum_{\alpha+\beta=d} |a_{\alpha,\beta}| \big\| (\cdot)^\alpha f_1 \big\|_{L^{p'}(Q)} \big\| C_b R_i((\cdot)^\beta f_2 1_Q) \big\|_{L^p(Q)} \\
&\le \sum_{i=1}^d \sum_{\alpha+\beta=d} |a_{\alpha,\beta}| \big\| (\cdot)^\alpha \big\|_{L^\infty(Q)} \big\| f_1 \big\|_{L^{p'}(Q)} \big\| (\cdot)^\beta \big\|_{L^\infty(Q)} \big\| C_b R_i \big\|_{L^p\to L^p} \big\| f_2 \big\|_{L^p(Q)} \\
&\le C_d \sum_{i=1}^d \sum_{\alpha+\beta=d} |a_{\alpha\beta}| |Q| \big\| C_b R_i \big\|_{L^p\to L^p} \big\| f_1 \big\|_{L^{p'}(Q)} \big\| f_2 \big\|_{L^p(Q)} \\
&\le C_d |Q| \sum_i^d \big\| C_b R_i \big\|_{L^p\to L^p} \big\| f_1 \big\|_{L^{p'}(Q)} \big\| f_2 \big\|_{L^p(Q)},
\end{aligned}$$

where at $*$ we used the fact that the limits exist separetely as $R_i((\cdot)^\beta 1_Q f_2 b(x,\cdot))(x)$, and where in the second to last esimate we used the assumption that Q is centered at the origin. Dividing by $|Q|^2$ gives the claim.

Case 2, $d \ge 2$: In the previous estimate we saw that the key issue with the lower bound for $C_b R_i$'s is the following: We introduce 1 as $\sum_{i=1}^d (x_i-y_i)^2 |x-y|^{-2}$, and would like to view this as $(x_i-y_i)|x-y|^{-d-1}$ times functions that depend only on x and only on y. As we saw:

$$1 = \sum_{i=1}^d \frac{(x_i-y_i)^2}{|x-y|^2} = \sum_{i=1}^d \frac{(x_i-y_i)}{|x-y|^{d+1}} (x_i-y_i)|x-y|^{d-1}, \tag{2.2}$$

and the problem becomes about expanding $|x-y|^{d-1}$ when d is even, hence d was odd.

Consider the function $z_i|z|^{d-1}$ of $z \in \mathbb{R}^d$. By induction, we check that $\partial^\alpha(z_i|z|^{d-1})$ is a linear combination of terms of the form $z^\beta |z|^{d-|\alpha|-|\beta|}$, where $|\beta| \le |\alpha|+1$. In particular, when $|\alpha| = d+1$, then $\partial^\alpha(z_i|z|^{d-1})$ is a linear combination of terms of the form $z^\beta|z|^{-1-|\beta|}$. In particular, $|\partial^\alpha(z_i|z|^{d-1})| \lesssim |z|^{-1} \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ for $d \ge 2$.

Now let $\phi \in C_c^\infty(\mathbb{R}^d)$ be a smooth bump such that $\phi \equiv 1$ in $Q(0, \frac{1}{4})$, and ϕ is supported in $Q(0, \frac{1}{2})$ (cube of centre 0 and "radius" $\frac{1}{2}$, hence sidelength 1). We consider the function $\phi_i(z) = \phi(z) z_i |z|^{d-1}$. By the previous computation and product rule, this satisfies

$$|\partial^\alpha \phi_i| \lesssim |z|^{-1} 1_{Q(0,\frac{1}{2})} \in L^1(\mathbb{R}^d)$$

for $|\alpha| = d+1$ and $d \geq 2$.

Thus the Fourier transform of ϕ_i satisfies for all $|\alpha| = d+1$,

$$|k^\alpha \hat{\phi}_i(k)| \sim \Big| \int \partial_z^\alpha \phi_i(z) e^{-i2\pi k\cdot z} \, \mathrm{d}z \Big| \leq \int |\partial_z^\alpha \phi_i(z)| \, \mathrm{d}z < \infty,$$

and hence $|\hat{\phi}_i(k)| \lesssim |k|^{-1-d}$. If Φ_i is the 1-periodic extension of ϕ_i, its Fourier coefficients satisfy this same estimate. In particular, these Fourier coefficient are in $\ell^1(\mathbb{Z}^d)$. Recalling that $\phi_i(z)$ agrees with $z_i|z|^{d-1}$ in $Q(0, \frac{1}{4})$, we hence have shown that

$$z_i |z|^{d-1} = \sum_{k \in \mathbb{Z}^d} a_i(k) e^{i2\pi k\cdot z}, \qquad \forall z \in Q(0, \frac{1}{4}), \tag{2.3}$$

where $\sum_{k\in\mathbb{Z}^d} |a_i(k)| < \infty$.

And observe that we only need to apply the formula (2.2) when $x, y \in Q$, a given cube. By composing back and forth with dilations in addition to translations, we may assume that $Q = Q(0, \frac{1}{8})$. Then if $x, y \in Q$, we see that $x - y \in Q(0, \frac{1}{4})$, where (2.3) is valid. Substituting (2.3) with $z = x - y$ into (2.2), we obtain

$$1 = \sum_{i=1}^d \frac{(x_i - y_i)}{|x-y|^{d+1}} \sum_{k\in\mathbb{Z}^d} a_i(k) e^{i2\pi k\cdot(x-y)} = \sum_{k\in\mathbb{Z}^d} a_i(k) \sum_{i=1}^d \frac{(x_i - y_i)}{|x-y|^{d+1}} e^{i2\pi k\cdot x} e^{-i2\pi k\cdot y},$$

which is a convergent series of expressions of the desired form, namely the Riesz transform kernel multiplied by (bounded) functions that depend only on x or only on y. After this, the argument can be concluded in the same way as before.

This Fourier series idea is based on Svante Janson [8]. □

We gather two more basic estimates.

Lemma 2.3. *Let Q be a cube and $b_i \in L^3_{loc}$, $i = 1, 2$, be such that $\fint_Q b_i = 0$. Then*

$$\Big| \fint_Q \fint_Q (b_1(x) - b_1(y))(b_2(x) - b_2(y)) \overline{b_1(x) b_2(y)} \, \mathrm{d}y \, \mathrm{d}x \Big| \geq \fint_Q |b_1|^2 \fint_Q |b_2|^2 \tag{2.4}$$

and

$$\fint_Q \fint_Q (b_1(x) - b_1(y))(b_2(x) - b_2(y)) \overline{b_1(x) b_2(x)}) \, \mathrm{d}y \, \mathrm{d}x = \fint_Q |b_1 b_2|^2 + \Big| \fint_Q b_1 b_2 \Big|^2, \tag{2.5}$$

where we have replaced the latter occurrance of $b_2(y)$ with $b_2(x)$.

Proof. Multiplying out shows that

$$\begin{aligned}
&\fint_Q \fint_Q (b_1(x) - b_1(y))(b_2(x) - b_2(y)) \overline{b_1(x) b_2(y)} \, \mathrm{d}y \, \mathrm{d}x \\
&= \fint_Q b_1(x) b_2(x) \overline{b_1(x)} \, \mathrm{d}x \fint_Q \overline{b_2(y)} \, \mathrm{d}y - \fint_Q b_1(x) \overline{b_1(x)} \, \mathrm{d}x \fint_Q b_2(y) \overline{b_2(y)} \, \mathrm{d}y \\
&- \fint_Q b_2(x) \overline{b_1(x)} \, \mathrm{d}x \fint_Q b_1(y) \overline{b_2(y)} \, \mathrm{d}y + \fint_Q \overline{b_1(x)} \, \mathrm{d}x \fint_Q b_1(y) b_2(y) \overline{b_2(y)} \, \mathrm{d}y \\
&= -\fint_Q |b_1(x)|^2 \, \mathrm{d}x \fint_Q |b_2(y)|^2 \, \mathrm{d}y - \Big| \fint_Q b_1(x) \overline{b_2(x)} \, \mathrm{d}x \Big|^2,
\end{aligned}$$

whence

$$\begin{aligned}&\Big|\fint_Q\fint_Q(b_1(x)-b_1(y))(b_2(x)-b_2(y)\overline{b_1(x)b_2(y)}\,\mathrm{d}y\,\mathrm{d}x\Big|\\&=\fint_Q|b_1(x)|^2\,\mathrm{d}x\fint_Q|b_2(y)|^2\,\mathrm{d}y+\Big|\fint_Q b_1(x)\overline{b_2(x)}\,\mathrm{d}x\Big|^2\geq\fint_Q|b_1(x)|^2\,\mathrm{d}x\fint_Q|b_2(y)|^2\,\mathrm{d}y.\end{aligned}$$

As for (2.5) we compute:

$$\begin{aligned}&\fint_Q\fint_Q(b_1(x)-b_1(y))\,(b_2(x)-b_2(y))\,\overline{b_1(x)b_2(x)}\,\mathrm{d}y\,\mathrm{d}x\\&=\fint_Q|b_1b_2|^2-\fint_Q b_2\fint_Q|b_1|^2\overline{b_2}-\fint_Q b_1\fint_Q\overline{b_1}|b_2|^2+\fint_Q b_1b_2\fint_Q\overline{b_1b_2}\\&=\fint_Q|b_1b_2|^2+\Big|\fint_Q b_1b_2\Big|^2.\end{aligned}$$

□

The lower bounds now follow by combining lemmas 2.2 and 2.3.

Theorem 2.4. *Let R_j, $j=1,\ldots,d$, be the Riesz transforms, $b_1b_2\in L^2_{loc}$, $b_1,b_2\in L^3_{loc}$. Then*

$$S_2(b_1,b_2)+T_2(b_1,b_2)\leq C_d\sum_{j=1}^d\big\|C_bR_j\big\|_{L^2\to L^2}.$$

Proof. Denote $\psi_i=b_i-\langle b_i\rangle_Q$, $i=1,2$. Then $\int_Q\psi_i=0$ and the assumptions of Lemma 2.3 are satisfied by which by (2.4) and lemma 2.2 we get the necessary condition S_2

$$\begin{aligned}\fint_Q|\psi_1(x)|^2\,\mathrm{d}x\fint_Q|\psi_2(y)|^2\,\mathrm{d}y&\leq\Big|\fint_Q\fint_Q(\psi_1(x)-\psi_1(y))(\psi_2(x)-\psi_2(y))\overline{\psi_2(y)\psi_1(x)}\,\mathrm{d}y\,\mathrm{d}x\Big|\\&\leq C_d\sum_{i=1}^k\big\|C_bR_i\big\|_{L^2\to L^2}\Big(\fint_Q|\psi_1|^2\Big)^{1/2}\Big(\fint_Q|\psi_2|^2\Big)^{1/2}.\end{aligned}$$

For the condition T_2, we apply lemma 2.2 with $f_2=1_Q$, $f_1=\overline{\psi_1\psi_2}$ and lemma 2.3 by (2.5) with $b_i=\psi_i$ to attain

$$\begin{aligned}&\fint_Q|\psi_1\psi_2|^2\leq\Big|\fint_Q\fint_Q(b_1(x)-b_1(y))\,(b_2(x)-b_2(y))\,\overline{\psi_1(x)\psi_2(x)}1_Q(y)\,\mathrm{d}y\,\mathrm{d}x\Big|\\&\leq C_d\sum_{i=1}^k\big\|C_bR_i\big\|_{L^p\to L^p}\Big(\fint_Q|\psi_1\psi_2|^2\Big)^{1/2}\Big(\fint_Q|f_2|^2\Big)^{1/2}=C_d\sum_{i=1}^k\big\|C_bR_i\big\|_{L^p\to L^p}\Big(\fint_Q|\psi_1\psi_2|^2\Big)^{1/2}.\end{aligned}$$

Dividing out equal factors and summing gives the claim. □

3. Sufficient conditions

In this section we specify T to be a Calderón-Zygmund operator satisfying the Dini condition. We begin with partially recalling, with only minor modifications, a sparse domination of T from Lerner [9] (see also [10]) and its commutators from Ibánez-Firnkorn–Rivera-Ríos [7]. The sparse domination would quickly give the boundedness of the commutator C_bT on L^2, whenever $S_p(b_1,b_2)+T_p(b_1,b_2)<\infty$ for any $p>2$. However, in the last section we find that this is too strong to characterize the boundedness of C_bT on L^2 and hence are motivated to introduce the condition $S_{A,B}+T_C$ involving the Young functions A,B,C, that can be made strictly weaker than S_p+T_p for all $p>2$. Lastly, we prove the upper bound in Theorem 3.10 with these updated conditions.

We begin with definitions.

Definition 3.1. A d-dimensional Calderón-Zygmund operator T with an ω-Dini -kernel is a $L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ bounded operator with the representation

$$Tf(x) = \int K(x,y)f(y)\,\mathrm{d}y, \qquad x \notin \operatorname{spt}(f),$$

with the kernel $K : \mathbb{R}^d \times \mathbb{R}^d \setminus \{(x,x) : x \in \mathbb{R}^d\} \to \mathbb{C}$ satisfying the size condition $|K(x,y)| \leq C|x-y|^{-d}$ and the regularity condition

$$|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \leq \frac{C'}{|x-y|^d}\omega\left(\frac{|x-x'|}{|x-y|}\right),$$

whenever $|x-x'| \leq \frac{1}{2}|x-y|$, with the modulus of continuity $\omega\colon [0,1] \to \mathbb{R}_+$ that is continuous, increasing, subadditive, satisfies $\omega(0) = 0$ and $\|w\|_{\mathrm{Dini}} := \int_0^1 \omega(t)\frac{dt}{t} < \infty$.

Definition 3.2. Given a $\gamma \in (0,1)$, we say that a collection of sets $\mathcal{F}$ is γ-sparse, if for all distinct elements $S, R \in \mathcal{F}$, there exist sets $E_S \subset S, E_R \subset R$ such that $E_S \cap E_R = \emptyset$ and $|E_S| > \gamma|S|$.

Definition 3.3. Let T be as in definition 3.1. We have the following maximal operators on $L^2(\mathbb{R}^d)$:

i) the maximal operator $T_*f(x) = \sup_{\varepsilon>0} \left|Tf1_{B(x,\varepsilon)^c}(x)\right|$,
ii) the grand maximal operator $\mathcal{M}_T(f)(x) = \sup_{Q\ni x} \operatorname{ess\,sup}_{\xi\in Q} \left|T(f1_{\mathbb{R}^d\setminus 3Q})(\xi)\right|$,
iii) and its localized version $\mathcal{M}_{T,Q}(f)(x) = \sup_{Q\supset P\ni x} \operatorname{ess\,sup}_{\xi\in P} \left|T(f1_{3Q\setminus 3P})(\xi)\right|$, where Q, P are cubes.

The control over the grand maximal operator is given by

Lemma 3.4. [9, Lemma 3.2] *Let $f \in L^2_{loc}$. The following pointwise estimates hold:*

i) for a.e. $x \in Q$ we have: $\left|T(f1_{3Q}(x))\right| \leq C_d\left\|T\right\|_{L^1\to L^{1,\infty}}|f(x)| + \mathcal{M}_{T,Q}f(x)$,
ii) for all $x \in \mathbb{R}^d$ we have: $\mathcal{M}_Tf(x) \leq C_d(\left\|\omega\right\|_{\mathrm{Dini}} + C_T)\mathcal{M}f(x) + T_*f(x)$.

For a more refined argument for the sparse domination in Theorem 3.5 without Lemma 3.4, see the latest version of the sparse domination principle in Lerner, Ombrosi [10].

Theorem 3.5. *Let T be a d-dimensional Calderón-Zygmund operator with a Dini kernel and denote $b(x,y) = (b_1(x) - b_1(y))(b_2(x) - b_2(y))$. We assume that $f \in L^1_c(\mathbb{R}^d)$, and further to make everything well-defined that $b_1, b_2, b_1b_2, b_1f, b_2f, b_1b_2f \in L^1_{loc}$.*

From these assumptions it follows that there exists a sparse collection S of cubes on $\mathbb{R}^d$ such that

$$\left|C_bTf(x)\right| \leq C_{T,d}\sum_{i=1}^{4} S_if(x),$$

where

$$S_1f = \sum_{Q\in S} |b_1 - \langle b_1\rangle_Q||b_2 - \langle b_2\rangle_Q|\big\langle|f|\big\rangle_Q 1_Q, \qquad S_2f = \sum_{Q\in S} |b_2 - \langle b_2\rangle_Q|\big\langle|b_1 - \langle b_1\rangle_Q||f|\big\rangle_Q 1_Q,$$

$$S_3f = \sum_{Q\in S} |b_1 - \langle b_1\rangle_Q|\big\langle|b_2 - \langle b_2\rangle_Q||f|\big\rangle_Q 1_Q, \qquad S_4f = \sum_{Q\in S} \big\langle|b_1 - \langle b_1\rangle_Q||b_2 - \langle b_2\rangle_Q||f|\big\rangle_Q 1_Q,$$

and the sparse constant denoted with γ depends only on the dimension d.

Proof. We recall only the part of the proof where the exceptional set is defined and control over the appearing terms is established. In addition, a comment is made about the rest of the proof, the details for which we refer the reader to the proof of Theorem 1.1 in [12] or [9].

For an arbitrary integrable function $\psi \neq 0$ on Q define

$$E_1(\psi) = \{x \in Q\colon |\psi(x)| > \alpha\langle|\psi|\rangle_{3Q}\}, \qquad E_2(\psi) = \{x \in Q\colon \mathcal{M}_{T,Q}\psi(x) > \alpha\langle|\psi|\rangle_{3Q}\}$$

and let the exceptional set be

$$E = \bigcup_{i=1,2} E_i(f) \cup E_i(b_1f) \cup E_i(b_2f) \cup E_i(b_1b_2f).$$

Since the localized version of the grand maximal operator is controlled with the non-localized by

$$\mathcal{M}_{T,Q}f \leq \mathcal{M}_T(f1_{3Q}),$$

and by the well-known facts that $\mathcal{M}, T_*\colon L^1 \to L^{1,\infty}$ boundedly, it follows from the weak $(1,1)$ bounds implied by ii) of Lemma 3.4 in conjunction with the local integrability of all functions in question that we may choose some $\alpha > 0$ independent of the cube Q so that $|E| \leq 2^{-(d+2)}|Q|$.

Taking a Calderón-Zygmund decomposition of the function 1_E at the height $2^{-(d+1)}$ yields a collection $\mathcal{F}$ of cubes satisfying:

$$\sum_{P\in\mathcal{F}} |P| \leq \frac{1}{2}|Q|, \quad |E \setminus \bigcup_{P\in\mathcal{F}} P| = 0 \quad \text{and} \quad P \cap E^c \neq \emptyset \qquad \forall P \in \mathcal{F}.$$

Then one decomposes

$$(C_bT(f1_{3Q}))\, 1_Q = (C_bT(f1_{3Q}))\, 1_{Q\setminus\cup P} + \sum_{P\in\mathcal{F}} \big(C_bT(f1_{3Q\setminus 3P})\big)\, 1_P + \sum_{P\in\mathcal{F}} (C_bT(f1_{3P}))\, 1_P$$

and uses the properties of the collection $\mathcal{F}$, Lemma 3.4 and that the commutator is unchanged modulo constants in the functions b_1, b_2 to derive

$$\begin{aligned}\Big|C_bT(f1_{3Q})\Big| 1_Q \leq C_{T,d}\Big(&|b_2 - \langle b_2\rangle_{3Q}||b_1 - \langle b_1\rangle_{3Q}|\langle |f|\rangle_{3Q} \\ &+ |b_2 - \langle b_2\rangle_{3Q}|\langle |b_1 - \langle b_1\rangle_{3Q}||f|\rangle_{3Q} \\ &+ |b_1 - \langle b_1\rangle_{3Q}|\langle |b_2 - \langle b_2\rangle_{3Q}||f|\rangle_{3Q} \\ &+ \langle |b_1 - \langle b_1\rangle_{3Q}||b_2 - \langle b_2\rangle_{3Q}||f|\rangle_{3Q}\Big)1_Q + \sum_{P\in\mathcal{F}} \Big|C_bT(f1_{3P})\Big| 1_P.\end{aligned}$$

From this situation one first iterates the above estimate with the last term and then transfers the limit construction from the local to the global. □

Before stating and proving theorem 3.10 we need to recall and define

3.6. **Young functions, their basic properties and the conditions $S_{A,B}, T_C$.** We may also define joint conditions involving Young functions. A function $A : [0,\infty) \to [0,\infty)$ is called a Young function if it is continuous, convex, strictly increasing and satisfies

$$A(0) = 0, \qquad \lim_{t\to\infty} A(t)/t = \infty.$$

Given a Young function A, the complementary Young function $\bar{A}$ is defined by

$$\bar{A}(t) = \sup_{s>0}\{st - A(s)\}, \quad t > 0.$$

We also have the maximal function associated with a Young function A :

$$M_Af(x) = \sup_{Q\ni x}\langle |f|\rangle_{A,Q},$$

where the Luxemburg norm is defined by

$$\langle |f|\rangle_{A,Q} = \inf\{\lambda > 0 : \frac{1}{|Q|}\int_Q A(|f|/\lambda) \leq 1\}.$$

We say that $f \in L^A_{\text{loc}}$ if $\langle |f|\rangle_{A,Q} < \infty$ for all cubes Q. The relative sizes of Young functions A, B are compared with the symbol $\succeq$; we say that $B \succeq A$, if there exist constants $C, t_0 > 0$ such that $A(t) \leq CB(t)$, when $t > t_0$. Finally, we define the B_p class: a Young function $A \in B_p$ for $p > 1$ if

$$\int_1^\infty \frac{A(t)}{t^p}\frac{dt}{t} < \infty.$$

We record the following properties, which can be found at least in [3, Chapter 5] (see also [15]):

Proposition 3.7. Given a Young function A, it holds that

i) for any $t \geq 0$, $t \leq A^{-1}(t)\bar{A}^{-1}(t) \leq 2t$,

ii) for any cube Q,

$$\langle|fg|\rangle_Q \le 2\langle|f|\rangle_{A,Q}\langle|g|\rangle_{\bar{A},Q}. \tag{3.1}$$

More generally, if A, B, and C are Young functions such that for all $t \ge t_0 > 0$,

$$B^{-1}(t)C^{-1}(t) \le cA^{-1}(t),$$

then

$$\langle|fg|\rangle_{A,Q} \lesssim \langle|f|\rangle_{B,Q}\langle|g|\rangle_{C,Q},$$

iii) if $B \succeq A$, then $\langle|f|\rangle_{A,Q} \lesssim \langle|f|\rangle_{B,Q}$ and $M_A \lesssim M_B$,

iv) if $\bar{A} \in B_{p'}$, then $A(t) \succeq t^p$ and $\langle|f|^p\rangle_Q^{\frac{1}{p}} \lesssim \langle|f|\rangle_{A,Q}$.

Proposition 3.8. [15] $M_A : L^p \to L^p$ boundedly if and only if $A \in B_p$.

Now we are ready to give the following definition:

Definition 3.9. Given Young functions A, B, C such that $\bar{B}, \bar{C} \in B_p$, $\bar{A} \in B_{p'}$ and a pair of complex valued functions $b_1 \in L^A_{\mathrm{loc}}(\mathbb{R}^d), b_2 \in L^B_{\mathrm{loc}}(\mathbb{R}^d)$, we say that the joint condition $S_{A,B}$ holds if

$$S_{A,B}(b_1,b_2) := \sup_Q \big\langle |b_1 - \langle b_1\rangle_Q|\big\rangle_{A,Q}\big\langle |b_2 - \langle b_2\rangle_Q|\big\rangle_{B,Q} < \infty,$$

and for $b_1^2, b_2^2 \in L^C_{\mathrm{loc}}(\mathbb{R}^d)$, we say that the joint condition T_C holds if

$$T_C(b_1,b_2) := \sup_Q \big\langle |b_1 - \langle b_1\rangle_Q||b_2 - \langle b_2\rangle_Q|\big\rangle_{C,Q} < \infty. \tag{3.2}$$

We remark immediately, that in Theorem 4.5 we find a commutator that is unbounded on L^2 and that satisfies the conditions $S_2 + T_2$ but fails the conditions $S_{A,B} + T_C$ for all Young functions $\bar{A}, \bar{B}, \bar{C} \in B_2$.

Theorem 3.10. *Assume that a pair of functions $b_1 \in L^A_{\mathrm{loc}}(\mathbb{R}^d)$ and $b_2 \in L^B_{\mathrm{loc}}(\mathbb{R}^d)$ with $b_1^2, b_2^2 \in L^C_{\mathrm{loc}}(\mathbb{R}^d)$ satisfy the conditions T_C and $S_{A,B}$ for some Young functions A, B, C with $\bar{A}, \bar{B}, \bar{C} \in B_2$, then*

$$S_i \colon L^2(\mathbb{R}^d) \cap L^3_c(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d), \qquad i = 1,2,3,4$$

boundedly.

Especially, it follows with a standard density argument by Theorem 3.5 that

$$C_bT \colon L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d)$$

boundedly when notation and assumptions are retained.

Proof. The pairs of terms S_1, S_4 and S_2, S_3 are symmetric with respect to dual pairings. Hence, we show the estimate in the two distinct cases of S_1 and S_3. By duality it is enough to estimate the pairings $\langle S_i(f), \psi\rangle$.

First, for the term S_1 we only use the assumptions involving the functions A, B. By sparseness we get

$$\begin{aligned}
\Big|\langle S_1(f),\psi\rangle\Big| &\le \sum_{Q\in S}\int_Q |b_1 - \langle b_1\rangle_Q||\psi|\langle|b_2 - \langle b_2\rangle_Q||f|\rangle_Q \\
&\lesssim \sum_{Q\in S}|Q|\langle|b_1 - \langle b_1\rangle|\rangle_{A,Q}\langle|\psi|\rangle_{\bar{A},Q}\langle|b_2 - \langle b_2\rangle_Q|\rangle_{B,Q}\langle|f|\rangle_{\bar{B},Q} \\
&\le S_{A,B}(b_1,b_2)\sum_{Q\in S}|Q|\langle|\psi|\rangle_{\bar{A},Q}\langle|f|\rangle_{\bar{B},Q} \le \gamma^{-1}S_{A,B}(b_1,b_2)\sum_{Q\in S}\int_{E_Q}\langle|\psi|\rangle_{\bar{A},Q}\langle|f|\rangle_{\bar{B},Q} \\
&\le \gamma^{-1}S_{A,B}(b_1,b_2)\sum_{Q\in S}\int_{E_Q} M_{\bar{A}}\psi M_{\bar{B}}f \le \gamma^{-1}S_{A,B}(b_1,b_2)\big\|M_{\bar{A}}\psi\big\|_{L^2}\big\|M_{\bar{B}}f\big\|_{L^2} \\
&\lesssim S_{A,B}(b_1,b_2)\big\|\psi\big\|_{L^2}\big\|f\big\|_{L^2},
\end{aligned}$$

where we have used Proposition 3.8 in the last step.

Next we use the condition T_C to control the term S_3:

$$\begin{aligned}\left|\langle S_3(f),\psi\rangle\right| &\leq \sum_{Q\in S}|Q|\langle|\psi|\rangle_Q\langle|b_1-\langle b_1\rangle_Q||b_2-\langle b_2\rangle_Q||f|\rangle_Q\\ &\leq \sum_{Q\in S}|Q|\langle|\psi|\rangle_Q\langle|f|\rangle_{\bar{C},Q}\langle|b_1-\langle b_1\rangle_Q||b_2-\langle b_2\rangle_Q|\rangle_{C,Q}\\ &\leq T_C(b_1,b_2)\sum_{Q\in S}|Q|\langle|\psi|\rangle_Q\langle|f|\rangle_{\bar{C},Q}\leq \gamma^{-1}T_C(b_1,b_2)\big\|M\psi\big\|_{L^2}\big\|M_{\bar{C}}f\big\|_{L^2}\\ &\lesssim T_C(b_1,b_2)\big\|\psi\big\|_{L^2}\big\|f\big\|_{L^2}.\end{aligned}$$

□

Since with $A(t)=t^p$, $\bar{A}\in B_2$, for $p>2$, we immediately get:

Corollary 3.11. Let T be as before and assume that a pair of functions $b_1,b_2\in L^{2p}_{loc}(\mathbb{R}^d)$ satisfy the conditions T_p and S_p for some $p>2$. Then we have

$$C_bT\colon L^2(\mathbb{R}^d)\longrightarrow L^2(\mathbb{R}^d)$$

boundedly.

We close this section with some remarks.

Remark 3.12. For Theorem 3.10 the difference in the case $p\neq 2$ is that we need to introduce 3 more Young functions to manage the now non-symmetric dual pairings from the terms S_2,S_4. According to Definition 3.9 the existing Young functions functions are replaced with ones satisfying

$$\bar{A}\in B_{p'},\quad \bar{B},\bar{C}\in B_p$$

and are supplemented with Young functions D,E,F satisfying

$$\bar{D},\bar{F}\in B_p,\quad \bar{E}\in B_{p'}$$

and $S_{D,E}(b_1,b_2)+T_F(b_1,b_2)<\infty$.

Remark 3.13. Given a $q\in(2,\infty)$, adapting the proof of Theorem 3.10 shows that if b_1,b_2 satisfy the conditions $S_{q+\varepsilon},T_{q+\varepsilon}$ for any $\varepsilon>0$, then $C_bT:L^q\to L^q$ boundedly.

On the other hand, for $q\in(1,2)$, the conditions S_p,T_p with $p\in(q,2)$ are not strong enough to conclude that $C_bT:L^q\to L^q$ boundedly. Indeed, if they were, then by duality and interpolation $C_bT:L^2\to L^2$ boundedly and Theorem 2.4 would imply the condition S_2. This gives a contradiction since by Proposition 4.1 (see below) there exist functions ϕ,ψ such that S_p,T_p are satisfied and S_2 is not.

4. Conjecture and related examples

In this last section we continue discussing the conditions $S_{A,B},T_C$ and their interdependence with the boundedness properties of the commutator on different L^p spaces.

First, we note that it follows by the John-Nirenberg inequality that if $b_1,b_2\in\mathrm{BMO}$, then the conditions S_p,T_q hold for all $p,q\geq 1$. Hence, a natural question is immediate: Are S_p, and respectively T_p, equivalent for all or some $1\leq p<\infty$. Or even in a weaker sense: if both of the conditions S_p,T_p hold simultaneously, could we deduce that S_q or T_q holds for some $q>p$? By Theorem 4.3 the answer is no and the example located therein is the motivation for introducing joint conditions involving Young functions that can be made strictly weaker than $S_{2+\varepsilon}+T_{2+\varepsilon}$ for all $\varepsilon>0$.

The next proposition will clarify the situation and point out how the counterexample in Theorem 4.3 can be constructed. For this, recall, that a function $\omega:\mathbb{R}^d\to(0,\infty)$ is said to be in the class of A_p weights, $1<p<\infty$, if

$$[w]_{A_p}=\sup_Q\langle w\rangle_Q\langle w^{-\frac{p'}{p}}\rangle_Q^{\frac{p}{p'}}<\infty,\quad p'=\frac{p}{p-1},$$

where the supremum is taken over all cubes.

Proposition 4.1. Given $1 < p < q < \infty$, there exists functions $\phi, \psi \in L^p_{loc}$ satisfying the conditions S_p, T_p and failing the condition S_q.

Proof. Let
$$\psi(x) = x^{-\frac{2}{p+q}} 1_{(0,1)}, \qquad \phi = \psi^{-1} 1_{(0,1)}.$$
We check that the conditions S_p, T_p hold. Let $[a,b)$ be an arbitrary intervall such that $[a,b) \cap [0,1) = [c,d) \neq \emptyset$ (if the intersection is empty, then the claim is trivial). First,
$$\frac{1}{b-a}\int_a^b |\psi - \langle \psi\rangle_{[a,b)}|^p \frac{1}{b-a}\int_a^b |\phi - \langle \phi\rangle_{[a,b)}|^p \leq \frac{1}{d-c}2^{p+1}\int_c^d \psi^p \frac{1}{d-c}2^{p+1}\int_c^d \phi^p.$$
Then, by the fact (see Grafakos [4]) that $|x|^{-\frac{2p}{p+q}} \in A_2$, we have
$$\frac{1}{d-c}\int_c^d \psi^p \frac{1}{d-c}\int_c^d \phi^p \leq [|x|^{-\frac{2p}{p+q}}]_{A_2}.$$
It follows that $S_p(\psi,\phi) \lesssim 1$.

By the above estimates and $\phi\psi \leq 1$, it follows for an arbitrary interval I that
$$\frac{1}{|I|}\int_I |\psi - \langle\psi\rangle_I|^p |\phi - \langle\phi\rangle_I|^p \leq 4^p \frac{1}{|I|}\left(\int_I \psi^p\phi^p + \int_I \psi^p \langle\phi\rangle_I^p + \int_I \phi^p\langle\psi\rangle_I^p + \langle\psi\rangle_I^p\langle\phi\rangle_I^p\right) \lesssim 1.$$
Hence $T_p(b_1,b_2) < \infty$.

On the other hand by $-2q/(p+q) < -1$, the singularity in $\int_0^1 |\psi - \langle\psi\rangle_{[0,1)}|^q$ is not integrable, and by $\int_0^1 |\phi - \langle\phi\rangle_{[0,1)}|^q > 0$, we have $S_q(\psi,\phi) = \infty$. □

Remark 4.2. If one wishes to have $\psi, \phi \in L^\infty_{loc}$, say to have the joint conditions well-defined, Proposition (4.1) can be modified by considering multiple copies of the situation spread out through $\mathbb{R}$ and introducing the singularities in ψ's only gradually as is done in the next theorem.

Theorem 4.3. *There exist functions $\psi, \phi \in L^\infty_{loc}$ failing the condition $S_{2+\varepsilon}$ for all $\varepsilon > 0$, such that $[\phi, [\psi, H]] : L^2 \to L^2$ boundedly, where H is the Hilbert transform, i.e. the 1-dimensional Riesz transform.*

Remark 4.4. By Theorem 2.4 the L^2 boundedness implies that ψ, ϕ satisfy the conditions T_2, S_2.

Proof of Theorem 4.3. Let
$$\psi_0^k(x) = c_k x^{-\eta_k} 1_{(c_k^{6k}e^{-100k^2},1)}(x), \qquad \eta_k = \frac{1}{2+k^{-1}}, \qquad \phi_0(x) = x^{1/2}1_{(0,1)}(x),$$
where c_k depends on k and will be determined later. Let $\tau_h f(x) = f(x-h)$. Then set $\phi_k = \tau_k\phi_0$ and $\psi_k = \tau_k\psi_0^k$. Finally we define
$$\phi = \sum_{k\in 2\mathbb{Z}} \phi_k, \qquad \psi = \sum_{k \in 4\mathbb{N}+2} \psi_k.$$
Let $k \in 4\mathbb{N}+2$ be fixed. We first show that the pair (ψ_k, ϕ) satisfies S_q, T_q for $q = q_k = 2 + k^{-1}/2$. Since $\tau_k\phi = \phi$ it suffices to prove that (ψ_0^k, ϕ) satisfies S_q, T_q. Again, for any interval I, we have
$$\frac{1}{|I|}\int_I |\psi_0^k - \langle\psi_0^k\rangle_I|^q \frac{1}{|I|}\int_I |\phi - \langle\phi\rangle_I|^q \leq 4^{q+1}\langle|\psi_0^k|^q\rangle_I\langle|\phi|^q\rangle_I.$$
We first consider the case when $\ell(I) \leq 1$ and we may further assume that $I \subset (0,1)$. Since $q\eta_k < 1$ we know that $|x|^{-q\eta_k} \in A_2$ and hence, by $I \subset (0,1)$,
$$4^{q+1}\langle|\psi_0^k|^q\rangle_I\langle|\phi|^q\rangle_I \leq 4^{7/2}c_k^{5/2}[|x|^{-q\eta_k}]_{A_2}.$$
It remains to consider the case when $\ell(I) > 1$. Since certainly $(0,1)\cap I \neq \emptyset$ (as otherwise there is nothing to prove) we know that $(0,1) \subset 3I$. Then due to that ϕ is a periodic function we have
$$4^{q+1}\langle|\psi_0^k|^q\rangle_I\langle|\phi|^q\rangle_I \leq 4^{q+1}\langle|\psi_0^k|^q\rangle_{(0,1)}\langle|\phi|^q\rangle_{(0,1)} \leq 4^{7/2}c_k^{5/2}[|x|^{-q\eta_k}]_{A_2}.$$

Therefore, we conclude that

$$S_q \leq 4^{7/2} c_k^{5/2} [|x|^{-q\eta_k}]_{A_2}.$$

On the other hand, since $\psi_0^k \phi_0 \leq c_k$, then by similar arguments as in Proposition 4.1 we have

$$T_q \leq 4^{7/2} c_k^{5/2} [|x|^{-q\eta_k}]_{A_2}.$$

Hence by Theorem 3.10 (see below) we know that the commutator $[\phi, [\psi_k, H]]$ is bounded on L^2 with norm $\sim c_k^{5/2} [|x|^{-q\eta_k}]_{A_2} \|M_{q'}\|^2_{L^2 \to L^2}$. Thus, we may further demand the constant c_k to be so small that $\big\|[\phi, [\psi_k, H]]\big\|_{L^2 \to L^2} \leq 2^{-k}$. Then $[\phi, [\psi, H]]$ also is bounded on L^2:

$$\big\|[\phi, [\psi, H]]\big\|_{L^2 \to L^2} = \big\|[\phi, [\sum_{k \in 4\mathbb{N}+2} \psi_k, H]]\big\|_{L^2 \to L^2} \leq \sum_{k \in 4\mathbb{N}+2} \big\|[\phi, [\psi_k, H]]\big\|_{L^2 \to L^2} \leq \sum_{k=1}^{\infty} 2^{-k} = 1.$$

It remains to check that the pair (ψ, ϕ) is precisely what we need. It is obvious that $\psi, \phi \in L^\infty_{loc}$. It remains to verify that (ψ, ϕ) fails $S_{2+\varepsilon}$ for any $\varepsilon > 0$. By Hölder's inequality we can assume $0 < \varepsilon < 1$. Find $\ell \in \mathbb{N}$ such that with $k := 4l + 2$ it holds that $(2+\varepsilon)\eta_k > 1 + (2k)^{-1}$. Hence, with $I = (k, k+1)$ we get

$$\begin{aligned}\int_I \Big|\psi - \langle\psi\rangle_I\Big|^{2+\varepsilon} = \int_0^1 \Big|\psi_0^k - \langle\psi_0^k\rangle_{(0,1)}\Big|^{2+\varepsilon} &\geq c_k^{2+\varepsilon} \int_{c_k^{6k} e^{-100k^2}}^{2c_k^{6k} e^{-100k^2}} \Big| x^{-\frac{1}{2+k^{-1}}} - \int_{c_k^{6k} e^{-100k^2}}^1 x^{-\frac{1}{2+k^{-1}}} \Big|^{2+\varepsilon} \mathrm{d}x \\ &\gtrsim c_k^{2+\varepsilon} \int_{c_k^{6k} e^{-100k^2}}^{2c_k^{6k} e^{-100k^2}} \Big| x^{-\frac{1}{2+k^{-1}}} \Big|^{2+\varepsilon} \mathrm{d}x \\ &\gtrsim c_k^{\varepsilon - 1} e^{50k}.\end{aligned}$$

On the other hand,

$$\int_I \Big|\phi - \langle\phi\rangle_I\Big|^{2+\varepsilon} = \int_0^1 \Big|\phi_0 - \langle\phi_0\rangle_{(0,1)}\Big|^{2+\varepsilon} \sim 1.$$

We conclude the proof by letting $\ell \to \infty$. □

Theorem 4.5. *There exists $b_1, b_2 \in L^\infty_{\mathrm{loc}}(\mathbb{R})$ such that $S_2(b_1, b_2) + T_2(b_1, b_2) < \infty$, but $S_{A,B}(b_1, b_2) = \infty$ and $T_C(b_1, b_2) = \infty$ for arbitrary Young functions A, B, C with $\bar{A}, \bar{B}, \bar{C} \in B_2$. Moreover, $C_b H : L^2 \not\to L^2$.*

Proof. We prove the result via the following example. Let $I_0 = [-1, 1]$ and

$$\sigma = 1_{I_0}, \qquad w = M(\sigma)^{-1},$$

notice that both σ and w are even functions. It is immediate to see that

$$\sup_I \langle\sigma\rangle_I \langle w\rangle_I \leq \sup_I \inf_{x \in I} M(\sigma)(x) \langle w\rangle_I \leq \sup_I \langle M(\sigma) w\rangle_I = 1. \tag{4.1}$$

Now define

$$b_1(x) := \mathrm{sgn}(x)\sigma(x), \qquad b_2(x) := \mathrm{sgn}(x) w^{\frac{1}{2}}(x),$$

and notice that immediately $b_1, b_2 \in L^\infty_{\mathrm{loc}}$. By (4.1) we see that

$$S_2(b_1, b_2)^2 = \sup_I \big\langle |b_1 - \langle b_1\rangle_I|^2 \big\rangle_I \big\langle |b_2 - \langle b_2\rangle_I|^2 \big\rangle_I \leq 16 \sup_I \big\langle |b_1|^2 \big\rangle_I \big\langle |b_2|^2 \big\rangle_I \leq 16 < \infty. \tag{4.2}$$

We also have

$$|b_1 - \langle b_1\rangle_I|^2 |b_2 - \langle b_2\rangle_I|^2 \leq 4(|b_1|^2 + |\langle b_1\rangle_I|^2)(|b_2|^2 + |\langle b_2\rangle_I|^2),$$

and by $|b_1 b_2| \leq 1$, direct calculations give us

$$T_2(b_1, b_2)^2 = \sup_I \big\langle |b_1 - \langle b_1\rangle_I|^2 |b_2 - \langle b_2\rangle_I|^2 \big\rangle_I \leq 4 + 12 \big\langle |b_1|^2 \big\rangle_I \big\langle |b_2|^2 \big\rangle_I \leq 16 < \infty. \tag{4.3}$$

However, for $J_k = (-k, k)$, $k \geq 2$, since b_1 and b_2 are odd functions,

$$S_{A,B}(b_1, b_2) \geq \lim_{k \to \infty} \big\langle |b_1| \big\rangle_{A, J_k} \big\langle |b_2| \big\rangle_{B, J_k} \gtrsim \lim_{k \to \infty} \big\langle |b_1| \big\rangle_{A, J_k} \big\langle |b_2|^2 \big\rangle_{J_k}^{\frac{1}{2}}$$

$$\sim \lim_{k\to\infty} \frac{k^{\frac{1}{2}}}{A^{-1}(k)} \sim \lim_{k\to\infty} \frac{\bar{A}^{-1}(k)}{k^{\frac{1}{2}}} = \lim_{k\to\infty} \Big(\frac{(\bar{A}^{-1}(k))^2}{\bar{A}(\bar{A}^{-1}k)} \Big)^{\frac{1}{2}},$$

where we have used the fact that $M(1_{I_0})(x) \sim (1+|x|)^{-1}$. To conclude notice that immediately by definition $\lim_{t\to\infty} \bar{A}^{-1}(t) = \infty$ and

$$\lim_{t\to\infty} \frac{\bar{A}(t)}{t^2} \leq \frac{4}{\log 2} \lim_{t\to\infty} \int_t^{2t} \frac{\bar{A}(s)}{s^2} \frac{ds}{s} = 0.$$

On the other hand with $I_k = (0,k)$, $k \geq 100$, we have

$$\begin{aligned} T_C(b_1,b_2) &\geq \lim_{k\to\infty} \big\langle |b_1 - \langle b_1\rangle_{I_k}||b_2 - \langle b_2\rangle_{I_k}| \big\rangle_{C,I_k} \geq \lim_{k\to\infty} \big\langle |b_1 - \langle b_1\rangle_{I_k}||b_2 - \langle b_2\rangle_{I_k}| 1_{(0,1)} \big\rangle_{C,I_k} \\ &= \lim_{k\to\infty} (1-k^{-1}) \big\langle |b_2 - \langle b_2\rangle_{I_k}| 1_{(0,1)} \big\rangle_{C,I_k}. \end{aligned}$$

Since for $x > 1$, $b_2(x) = (\frac{x+1}{2})^{\frac{1}{2}}$ and for $0 < x \leq 1$, $b_2(x) = 1$, another direct calculation shows that

$$\langle b_2\rangle_{I_k} = \frac{\sqrt{2}}{3k} \big[(k+1)^{\frac{3}{2}} - 2\sqrt{2}\big]$$

by which and the assumption $k \geq 100$ we see that

$$|b_2 - \langle b_2\rangle_{I_k}| 1_{(0,1)} \geq ck^{\frac{1}{2}}.$$

Hence

$$T_{C,2}(b_1,b_2) \geq \lim_{k\to\infty} c(1-k^{-1}) k^{\frac{1}{2}} \langle 1_{(0,1)}\rangle_{C,I_k} = \lim_{k\to\infty} c(1-k^{-1}) \frac{k^{\frac{1}{2}}}{C^{-1}(k)} = \infty.$$

Next, we show that $C_bH : L^2 \not\to L^2$. To see this, let

$$f(x) = x^{-\frac{1}{2}} (\log x)^{-1} 1_{[100,\infty)}(x) \in L^2(\mathbb{R}).$$

We claim that $|C_bHf(x)| = \infty$ for all $x \in I_0$, and in showing this, hence conclude the unboundedness of C_bH on L^2. Indeed, since for $y \in [100,\infty)$

$$b_2(y) = (M(1_{I_0}))^{-\frac{1}{2}} = \sqrt{\frac{y+1}{2}},$$

and $b_1(x) = b_2(x) = \operatorname{sgn}(x)$, for any $x \in I_0$, we have

$$\begin{aligned} |C_bHf(x)| &= \Big| \int_{100}^{\infty} (b_1(x) - b_1(y))(b_2(x) - b_2(y)) \frac{f(y)}{x-y} dy \Big| = \Big| \int_{100}^{\infty} (\operatorname{sgn}(x) - \sqrt{\frac{y+1}{2}}) \frac{f(y)}{x-y} dy \Big| \\ &\sim \int_{100}^{\infty} \sqrt{\frac{y+1}{2}} \frac{f(y)}{y-x} dy \sim \int_{100}^{\infty} \frac{f(y)}{\sqrt{y}} dy = \infty. \end{aligned}$$

□

If we take $A(t) = B(t) = C(t) = t^{2+\varepsilon}$, where $\varepsilon > 0$, we immediately have the following

Corollary 4.6. The conditions S_2, T_2 holding simultaneously does not improve to $S_{2+\varepsilon}$ or $T_{2+\varepsilon}$ for any $\varepsilon > 0$.

For our next example, we note that functions $\Phi : [0,\infty) \to [0,\infty)$ of the form

$$\Phi(t) = t^p \log(e+t)^{p-1+\delta}, \qquad p \in (1,\infty),\ \delta \in (0,\infty)$$

are called log -bumps. These are Young functions, and we recall some facts from [3, Chapter 5]:

i) If $\Phi(t) = t^p \log(e+t)^{p-1+\delta}$, then

$$\Phi^{-1}(t) \sim t^{1/p} \log(e+t)^{-\frac{1}{p'} - \frac{\delta}{p}} \text{ and } \bar{\Phi}(t) \sim t^{p'} [\log(e+t)]^{-1-(p'-1)\delta} \in B_{p'}.$$

ii) If $\Phi(t) = t^p \log(e+t)^{p-1} \log\log(e^e+t)^{p-1+\delta}$(which is referred to as a loglog-bump), then
$$\Phi^{-1}(t) \sim t^{\frac{1}{p}} \log(e+t)^{-\frac{1}{p'}} \log\log(e^e+t)^{-\frac{1}{p'}-\frac{\delta}{p}}$$
and
$$\bar{\Phi}(t) \sim t^{p'} \log(e+t)^{-1} [\log\log(e^e+t)]^{-1-(p'-1)\delta} \in B_{p'}.$$

Theorem 4.7. *There exist functions $b_1, b_2 \in L^\infty_{loc}$ such that $C_bH : L^2 \to L^2$ boundedly, but $S_{A,B}(b_1, b_2) = \infty$ and $T_C(b_1, b_2) = \infty$, for all log -bumps A, B, C with $\bar{A}, \bar{B}, \bar{C} \in B_2$.*

Proof. The idea is to construct a pair of functions (b_1, b_2) such that it satisfies the assumption in Theorem 3.10 so that we can conclude the boundedness of C_bH directly, meanwhile, the related bump function increases slower than log-bumps. Let $\Phi_0 = t^2 \log(e+t) \log\log(e^e+t)^{3/2}$, and define
$$b_1(x) = \operatorname{sgn}(x) 1_{I_0}(x), \qquad b_2(x) = \operatorname{sgn}(x) \Phi_0^{-1}\big((M1_{I_0}(x))^{-1}\big), \ \ I_0 = [-1, 1].$$
We will show that (b_1, b_2) is what we need. First of all, it is easy to check that for any cube I,
$$\langle |b_1| \rangle_{\Phi_0, I} \langle |b_2| \rangle_{\Phi_0, I} \le \Big\langle \Phi_0^{-1}\big((M1_{I_0}(x))^{-1}\big)^{-1} |b_2| \Big\rangle_{\Phi_0, I} \le 1.$$
Then by the triangle inequality and general Hölder's inequality we have
$$\langle |b_1 - \langle b_1 \rangle_I| \rangle_{\Phi_0, I} \langle |b_2 - \langle b_2 \rangle_I| \rangle_{\Phi_0, I} \lesssim 1.$$
On the other hand, since $|b_1 b_2| \le 1$, using triangle inequality and general Hölder's inequality again we have
$$\langle |b_1 - \langle b_1 \rangle_I| |b_2 - \langle b_2 \rangle_I| \rangle_{\Phi_0, I} \lesssim 1.$$
Using Theorem 3.10 we know that C_bH is bounded on L^2, thanks to $\bar{\Phi}_0 \in B_2$.

It remains to show that $S_{A,B}(b_1, b_2) = \infty$ and $T_C(b_1, b_2) = \infty$, for all log -bumps A, B, C with $\bar{A}, \bar{B}, \bar{C} \in B_2$. Without loss of generality we can assume that $A(t) = t^2 \log(e+t)^{1+\alpha}$, $B(t) = t^2 \log(e+t)^{1+\beta}$ and $C(t) = t^2 \log(e+t)^{1+\gamma}$, where $\alpha, \beta, \gamma > 0$. For $S_{A,B}(b_1, b_2)$ again we test with the interval $J_k = (-k, k)$ with $k \ge 2$. Since b_1 and b_2 are odd functions, we have
$$\begin{aligned} \langle |b_1 - \langle b_1 \rangle_{J_k}| \rangle_{A, J_k} \langle |b_2 - \langle b_2 \rangle_{J_k}| \rangle_{B, J_k} &= \langle |b_1| \rangle_{A, J_k} \langle |b_2| \rangle_{B, J_k} \\ &\simeq k^{-\frac{1}{2}} \log(e+k)^{\frac{1+\alpha}{2}} k^{\frac{1}{2}} \log(e+k)^{-\frac{1}{2}} \log\log(e^e+k)^{-\frac{3}{4}} \\ &\overset{k\to\infty}{\to} \infty. \end{aligned}$$
For $T_C(b_1, b_2)$, we test with the cube $I_k = (0, k)$, $k \ge 100$, we have
$$\begin{aligned} T_C(b_1, b_2) &\ge \lim_{k\to\infty} \big\langle |b_1 - \langle b_1 \rangle_{I_k}| |b_2 - \langle b_2 \rangle_{I_k}| \big\rangle_{C, I_k} \ge \lim_{k\to\infty} \big\langle |b_1 - \langle b_1 \rangle_{I_k}| |b_2 - \langle b_2 \rangle_{I_k}| 1_{(0,1)} \big\rangle_{C, I_k} \\ &= \lim_{k\to\infty} (1 - k^{-1}) \big\langle |b_2 - \langle b_2 \rangle_{I_k}| 1_{(0,1)} \big\rangle_{C, I_k} \\ &\simeq \lim_{k\to\infty} (1 - k^{-1}) k^{\frac{1}{2}} \log(e+k)^{-\frac{1}{2}} \log\log(e^e+k)^{-\frac{3}{4}} k^{-\frac{1}{2}} \log(e+k)^{\frac{1+\alpha}{2}} = \infty. \end{aligned}$$
□

Corollary 4.8. The conditions $S_{2+\varepsilon}, T_{2+\varepsilon}$ are not precise enough to yield a characterization of $C_bH : L^2 \to L^2$.

Proof. The iterated commutator C_bH of Theorem 4.7 is bounded on L^2. We show that the conditions $S_{2+\varepsilon}(b_1, b_2)$ and $T_{2+\varepsilon}(b_1, b_2)$ are not satisfied for any $\varepsilon > 0$. To see this, it is enough to notice that for all log -bumps A, B, C with $\bar{A}, \bar{B}, \bar{C} \in B_2$, we have $t^{2+\varepsilon} \succeq A(t), B(t), C(t)$ by which by Proposition 3.7 iv) and the estimates in Theorem 4.7 it follows that
$$\langle |b_1 - \langle b_1 \rangle_{J_k}|^{2+\varepsilon} \rangle_{J_k}^{\frac{1}{2+\varepsilon}} \langle |b_2 - \langle b_2 \rangle_{J_k}|^{2+\varepsilon} \rangle_{J_k}^{\frac{1}{2+\varepsilon}} \gtrsim \langle |b_1 - \langle b_1 \rangle_{J_k}| \rangle_{A, J_k} \langle |b_2 - \langle b_2 \rangle_{J_k}| \rangle_{B, J_k} \to \infty,$$
and
$$\langle |b_1 - \langle b_1 \rangle_{I_k}|^{2+\varepsilon} |b_2 - \langle b_2 \rangle_{I_k}|^{2+\varepsilon} \rangle_{I_k}^{\frac{1}{2+\varepsilon}} \gtrsim \langle |b_1 - \langle b_1 \rangle_{I_k}| |b_2 - \langle b_2 \rangle_{I_k}| \rangle_{C, I_k} \to \infty,$$
as $k \to \infty$, showing that $S_{2+\varepsilon}(b_1, b_2) = \infty$ and $T_{2+\varepsilon}(b_1, b_2) = \infty$. □

Corollary 4.9. The commutator of Theorem 4.7 is bounded on L^2 and unbounded on all L^p, $p \in (1,\infty) \setminus \{2\}$.

Proof. Let $p > 2$, $q \in (2,p)$ and $f(x) = x^{-1/q} 1_{[100,\infty)}(x) \in L^p$. For all $x \in [-1,1]$,

$$|C_b H f(x)| = \left| \int (b_1(x) - b_1(y))(b_2(x) - b_2(y)) \frac{f(y)}{x-y} \,\mathrm{d}y \right|$$

$$\sim \int_{100}^{\infty} \frac{y^{\frac{1}{2}}}{\log(e+y)^{\frac{1}{2}} \log\log(e^e+y)^{\frac{3}{4}}} \frac{y^{-\frac{1}{q}}}{y} \,\mathrm{d}y = \infty,$$

showing that $C_b H : L^p \not\to L^p$. It follows by duality that also $C_b H : L^{p'} \not\to L^{p'}$. □

Remark 4.10. Alternatively, we can prove Corollary 4.8 by Corollary 4.9. Indeed, if the conditions $S_{2+\varepsilon}(b_1,b_2), T_{2+\varepsilon}(b_1,b_2)$ hold for some $\varepsilon > 0$, then by Remark 3.13 we have $C_b H : L^q \to L^q$ boundedly for all $q \in (2, 2+\varepsilon)$, a contradiction with Corollary 4.9.

The above considerations lead us to conjecture:

Conjecture 4.11. With the functions b_1, b_2 subject to the same assumptions as those in Theorems 3.10 and 2.4, the boundedness of $[b_1,[b_2,H]]$ on $L^2(\mathbb{R})$ is equivalent with the existence of Young functions A, B, C with $\bar{A}, \bar{B}, \bar{C} \in B_2$ such that $S_{A,B}(b_1,b_2) + T_C(b_1,b_2) < \infty$.

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